New parallel algorithm for the calculation of importance measures

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Abstract. A lot of mathematical approaches are used in importance analysis, which permits to investigate influence of system component state changes on the system reliability or availability. One of these approaches is Logical Differential Calculus, in particular Direct Partial Boolean Derivatives. A new algorithm for the calculation of Importance Measures with application of Direct Partial Boolean Derivatives is proposed in this paper. This algorithm is developed based on parallel procedures.

Keywords: Importance measures, Direct Partial Boolean Derivatives, matrix procedures, parallel algorithm

1 Introduction

Consider a system of n components. From reliability point of view, the system and all its components can be in one of two possible states: functional (presented as 1) and failed (presented as 0). The mathematical dependency between the system state and states of its components can be defined by the structure function [1]:

$$\phi(x_1, x_2,\ldots, x_n) = \phi(\boldsymbol{x}): \{0, 1\}^n \to \{0, 1\}, \quad (1)$$

where x_i is the state of component i, for $i = 1,2,\ldots,n$, and $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$ is a vector of components states (state vector).

Every system component is characterized by probability p_i (represents the availability of component i) and probability q_i (defines its unavailability):

$$p_i = \Pr\{x_i = 1\}, \quad q_i = \Pr\{x_i = 0\}, \quad p_i + q_i = 1. \quad (2)$$

When the system structure function and availabilities of all system components are known, then system availability/unavailability can be computed as follows [2, 3]:

$$A = \Pr\{\phi(\boldsymbol{x}) = 1\}, \quad U = \Pr\{\phi(\boldsymbol{x}) = 0\}, \quad A + U = 1. \quad (3)$$

The availability is one of the most important characteristics of any system. It can also be used to compute other reliability characteristics, e.g. mean time to failure,

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mean time to repair, etc. [2, 3]. But they do not permit to identify the influence of individual system components on the proper work of the system. For this purpose, there exist other measures that are known as *Importance Measures* (IM). The IMs are used in part of reliability analysis that is known as importance analysis. The comprehensive study of these measures has been performed in work [4]. IMs have been widely used for identifying system weaknesses and supporting system improvement activities from design perspective. With the known values of IMs of all components, proper actions can be taken on the weakest component to improve system availability at minimal costs or effort.

There exist a lot of IMs, but the most often used are the *Structural Importance* (SI), Birnbaum's Importance (BI), *Criticality Importance* (CI) and *Fussell-Vesely Importance* (FVI) (Table 1).

Table 1. Basic Importance Measures

Importance Measure	Meaning
SI	The SI concentrates only on the topological structure of the system. It is defined as the relative number of situations in which a given component is critical for the system activity
BI	The BI of a given component is defined as the probability that the component is critical for the system work.
CI	The CI of a given component is calculated as the probability that the system failure has been caused by the component failure, given that the system is failed.
FVI	The FVI of a given component is defined as the probability that the component contributes to the system failure probability.

Different mathematical methods and algorithms can be used to calculate these indices. Ones of them are *Direct Partial Boolean Derivatives* (DPBDs) that have been introduced for importance analysis in paper [5]. In paper [1], the mathematical background of DPBDs application has been considered. But efficient algorithm for computation of DPBDs has not been proposed. In this paper, a new parallel algorithm for the calculation of a DPBD is developed.

This paper has the next structure organization. In section 2, the general definition of a DPBD is provided. The definition and calculation aspects of IMs (Table 1) based on DPBDs are considered in this section too. In section 3, the development of the new parallel algorithm for DPBD calculation is considered. This algorithm is developed by the transformation of initial definition of a DPBD into matrix form. Based on the matrix definition of the DPBD, the new parallel algorithm is proposed.

As alternative result for the new algorithm, algorithms in [6] can be considered. The authors of the paper [6] proposed algorithms for calculation of a DPBD based on the structure function representation by a Binary Decision Diagram (BDD) that includes parallel procedure too. But the algorithms in [6] need a special transformation of initial representation of the structure function into a BDD, and this increases the computation complexity.

2 Importance Analysis

2.1 Direct Partial Boolean Derivatives

A DPBD is a part of Logical Differential Calculus [1, 7, 8]. In analysis of Boolean functions, a DPBD allows identifying situations in which the change of a Boolean variable results the change of the value of Boolean function. In case of reliability analysis, the system is defined by the structure function (1) that is a Boolean function. Therefore, a DPBD can be used for the structure function analysis too. In terms of reliability analysis, a DPBD allows investigation the influence of a structure function variable (=component state) change on a function value change (=system state). Therefore, a DPBD of the structure function permits indicating components states (state vectors) for which the change of one component state causes a change of the system state (availability). These vectors agree with the system boundary states [1, 5].

DPBD $\partial\phi(j \to \bar{j})/\partial x_i(a \to \bar{a})$ of the structure function $\phi(\boldsymbol{x})$ with respect to variable x_i is defined as follows [8]:

$$\frac{\partial\phi(j \to \bar{j})}{\partial x_i(a \to \bar{a})} = \{\phi(a_i,\boldsymbol{x}) \leftrightarrow j\} \wedge \{\phi(\bar{a}_i,\boldsymbol{x}) \leftrightarrow \bar{j}\}, \tag{4}$$

where $\phi(a_i, \boldsymbol{x}) = \phi(x_1, x_2,\ldots, x_{i-1}, a, x_{i+1},\ldots, x_n)$, $a, j \in \{0, 1\}$ and $\leftrightarrow$ is the symbol of equivalence operator (logical bi-conditional).

Clearly, there exist four DPBDs for every variable x_i [1, 7, 8]:

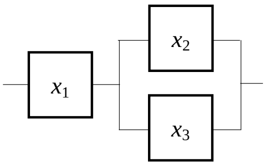
$$\frac{\partial\phi(1\to 0)}{\partial x_i(1\to 0)},\ \frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)},\ \frac{\partial\phi(1\to 0)}{\partial x_i(0\to 1)},\ \text{and}\ \frac{\partial\phi(0\to 1)}{\partial x_i(1\to 0)}.$$

In reliability analysis, the first two DPBDs can be used to identify situations in which a failure (repair) of component *i* results system failure (repair). Similarly, the second two DPBDs identify situations when the system failure (repair) is caused by the *i*-th component repair (failure). The second two derivatives exist (are not equal to zero) for a noncoherent systems [1]. In this paper, coherent systems are taken into account only. These systems meet the next assumptions [4]: (*i*) the structure function is monotone, and (*ii*) all components are independent and relevant to the system.

For example, consider a system of three components ($n = 3$) in Fig. 1 with structure function:

$$\phi(\boldsymbol{x}) = \mathrm{AND}(x_1, \mathrm{OR}(x_2, x_3)). \tag{5}$$

The influence of the first component failure on the system can be analyzed by DPBD $\partial\phi(1\to 0)/\partial x_1(1\to 0)$. This derivative has three nonzero values for state vectors $\boldsymbol{x} = (x_1, x_2, x_3)$: ($\underline{1\to 0}$, 1, 1), ($\underline{1\to 0}$, 0, 1) and ($\underline{1\to 0}$, 1, 0). Therefore, the failure of the first component causes a system breakdown for working state of the second and the third component or working state of one of them. The system is not functioning if the second and the third components are failed and, therefore, a failure of the first component does not influence system availability.



The structure function

$x_2\ x_3$ \ x_1	0	1
0 0	0	0
0 1	0	1
1 0	0	1
1 1	0	1

Fig. 1. The system example with its structure function

2.2 Importance Measures and Direct Partial Boolean Derivatives

In reliability analysis, the structure function and the system components are used instead of the Boolean function and the Boolean variables, respectively. Using this coincidence, the authors of the papers [1] have developed techniques for analysis of influence of individual system components on system failure/functioning using DPBDs. Let us summarize the definitions of IMs (Table 1) for the system failure based on DPBDs.

The SI of component i is defined as the relative number of situations, in which the component is critical for system failure. Therefore, the SI of component i can be defined by DPBD $\partial\phi(1\to0)/\partial x_i(1\to0)$ as the relative number of state vectors for which the considered DPBD has nonzero values [1, 5]:

$$SI_i = \frac{\rho_i^{(1\to0)}}{2^{n-1}}, \tag{6}$$

where $\rho_i^{(1\to0)}$ is a number of nonzero values of DPBD $\partial\phi(1\to0)/\partial x_i(1\to0)$ and 2^{n-1} is a size of the DPBD.

Similarly, the modified SI, which takes into account the necessary condition for component being critical, can be defined as follows [1, 5]:

$$MSI_i = \frac{\rho_i^{(1\to0)}}{\rho_i}, \tag{7}$$

where ρ_i is a number of state vectors for which $\phi(1_i, \boldsymbol{x}) = 1$.

The BI of component i defines the probability that the i-th system component is critical for system failure. Using DPBDs, this IM can be defined as the probability that the DPBD is nonzero [1]:

$$BI_i = \Pr\{\partial\phi(1\to0)/x_i(1\to0)\leftrightarrow1\}. \tag{8}$$

A lot of IMs are based on the BI, e.g. the CI, Barlow-Proschan, Bayesian, redundancy, etc. For example, the CI is calculated as follows [4]:

$$CI_i = BI_i \cdot \frac{q_i}{U}, \quad (9)$$

where q_i is component state probability (1) and U is the system unavailability.

The FVI takes into account the contribution of a component failure to system failure [4]. This contribution is computed based on *Minimal Cut Sets* (MCSs), which correspond to minimal sets of components whose simultaneous failure result system failure. Using MCSs, the FVI of the i-th component is defined as the probability that at least one MCS containing component i is failed, given that the system is failed [4]:

$$FVI_i = \Pr\{MCS(x_i) \mid \phi(\boldsymbol{x}) = 0\} = \frac{\Pr\{MCS(x_i) \cap \phi(\boldsymbol{x}) = 0\}}{\Pr\{\phi(\boldsymbol{x}) = 0\}} = \frac{\Pr\{MCS(x_i)\}}{U}, \quad (10)$$

where $MCS(x_i)$ represents the event when at least one MCS that contains component i is failed.

MCSs can also be expressed in the form of state vectors. These vectors are known as *Minimal Cut Vectors* (MCVs). A state vector $\boldsymbol{x}$ is a MCV if $\phi(\boldsymbol{x}) = 0$ and $\phi(\boldsymbol{x}') = 1$ for any $\boldsymbol{x}' > \boldsymbol{x}$ [2, 9]. Based on the investigation in paper [9], the FVI can be defined using MCVs as follows:

$$FVI_i = \frac{\Pr\{\boldsymbol{x} \le MCV(x_i)\}}{U}, \quad (11)$$

where $MCV(x_i)$ is a set of all MCVs for which $x_i = 0$.

According to [9], a state vector $\boldsymbol{x}$ is a MCV, if every variable meets one of the following two conditions: (*a*) DPBD $\partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ has a nonzero value for state vector $\boldsymbol{x}$, or (*b*) DPBD $\partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ does not exist for this state vector.

The condition (*a*) agrees with the definition of a MCV that supposes that $\phi(\boldsymbol{x}) = 0$ and $\phi(\boldsymbol{x}') = 1$ for any $\boldsymbol{x}' < \boldsymbol{x}$. The condition (*b*) assumes that $x_i = 1$ and, therefore, DPBD $\partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ for the state vector $\boldsymbol{x}$ does not exist.

Therefore, DPBDs $\partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ with respect to every variable of the structure function are calculated for the indication of MCVs. The intersection of these derivatives identifies the state vectors that are MCVs.

Consider some computational aspects of MCVs calculation based on DPBDs. It is known that the DPBD with respect to variable x_i does not depend on this variable [1, 8]. The derivative $\partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ is defined only for state vectors $(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ and cannot be computed for state vectors that have the form of $(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$. According to conditions (*a*) and (*b*), analysis of non-existing values of the DPBD is supposed in the calculation of MCVs. Definition of DPBD (4) is transformed and non-existing values of DPBD will be marked using a special symbol "*":

$$\partial\phi(0\to 1)/\partial x_i(0\to 1)=\begin{cases}1 & \text{if } x_i=0 \text{ and } \phi(1_i,\boldsymbol{x})\neq\phi(0_i,\boldsymbol{x})\\ 0 & \text{if } x_i=0 \text{ and } \phi(1_i,\boldsymbol{x})=\phi(0_i,\boldsymbol{x})\\ * & \text{if } x_i\neq 0\end{cases}. \quad (12)$$

For example, consider the system of three components in Fig. 1. All derivatives $\partial\phi(0\to 1)/\partial x_i(0\to 1)$ for this structure function are calculated according to (12) in Table 2. There are two MCVs for this system: (0, 1, 1) and (1, 0, 0). These MCVs are identified based on the intersection of all DPBDs. The rule for the intersection of two DPBDs (12) with respect to two different variables x_i and x_j is defined in Table 3 [9].

Table 2. Calculation of MCVs using DPBDs (12)

x_1 x_2 x_3	$\partial\phi(0\to1)/\partial x_1(0\to1)$	$\partial\phi(0\to1)/\partial x_2(0\to1)$	$\partial\phi(0\to1)/\partial x_3(0\to1)$	**The intersection**
0 0 0	0	0	0	**0**
0 0 1	1	0	*	**0**
0 1 0	1	*	0	**0**
0 1 1	1	*	*	**1**
1 0 0	*	1	1	**1**
1 0 1	*	0	*	**0**
1 1 0	*	*	0	**0**
1 1 1	*	*	*	*

Table 3. Defining the intersection of two DPBDs (12)

Value of $\partial\phi(0\to1)/\partial x_i(0\to1)$	Value of $\partial\phi(0\to1)/\partial x_j(0\to1)$		
	*	0	1
*	*	0	1
0	0	0	0
1	1	0	1

To illustrate the calculation of all IMs using DPBDs consider the system in Fig. 1. Values of IMs for this system are computed in Table 4. According to these IMs, the first component has the most influence on the system failure from point of view of the system structure, because the values of the SI, MSI and BI are greatest for this component. The CI is maximal for the second and third components and, therefore, it indicates the first component as non-important taking into account the probability of failure of this component (it is minimal for this component, i.e. q_1 = 0.10). The FVIs implies that the second and third components contribute to system failure with the most probability.

So, DPBDs are one of possible mathematical approaches that can be used in importance analysis, and they allow us to calculate all often used IMs (Table 1). Mathematical background of its application for the definition of IM has been considered in papers [1, 5]. In this paper new algorithm for the calculation of DPBD based on a parallel procedure is developed.

Table 4. IMs for the system in Fig.1

Component	Probability of component state, p_i	SI_i	MSI_i	BI_i	CI_i	FVI_i
x_1	0.90	0.75	1.00	0.90	0.46	0.52
x_2	0.70	0.25	0.50	0.32	0.49	0.54
x_3	0.65	0.25	0.50	0.27	0.49	0.54

3 Parallel Algorithm for the Calculation of Direct Partial Boolean Derivatives

One of possible way for the formal development of parallel algorithms is transform mathematical background into matrix algebra. Therefore, consider DPBD (4) in matrix interpretation. As the first step in such transformation, the initial data (structure function) has to be presented as a vector or matrix.

The structure function is defined as a *truth vector* (Fig. 2) in matrix algorithm for calculation of DPBD. It is column of a truth table of function $\mathbf{X} = [x^{(0)}\ x^{(1)}\ \ldots\ x^{(2^n-1)}]^T$, where $x^{(i)}$ is value of a function $\phi(\boldsymbol{x})$ for state vector $\boldsymbol{x} = (x_1, x_2,\ldots, x_n) = (i_1, i_2,\ldots, i_n)$ $((i_1 i_2\ldots i_n)$ is binary representation of the parameter i, $1 \le i \le 2^n-1)$.

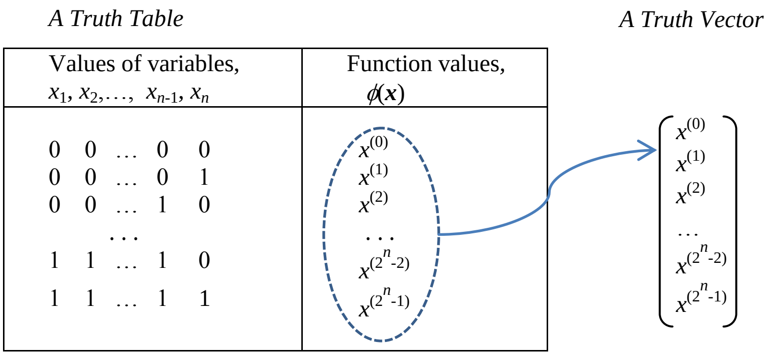


Fig. 2. Truth vector of the structure function

For example, the truth vector of the structure function for the system in Fig. 1 is:

$$\mathbf{X} = [x^{(0)}\ x^{(1)}\ x^{(2)}\ x^{(3)}\ x^{(4)}\ x^{(5)}\ x^{(6)}\ x^{(7)}]^T = [0\ 0\ 0\ 0\ 0\ 1\ 1\ 1]^T.$$

The value of the function can be defined by the truth vector unambiguously. Consider the truth vector element $x^{(5)} = 1$. The state vector for this function value is defined by the transformation of the parameter $i = 5$ into binary representation: $i = 5 \Rightarrow (i_1, i_2, i_3) = (1, 0, 1)$. Therefore, the truth vector element $x^{(5)} = 1$ agrees with the function value $\phi(1, 0, 1) = 1$.

The truth vector of DPBD (derivative vector) is calculated based on the truth vector of the structure function as:

$$\partial\mathbf{X}(j \to \bar{j})/\partial x_i(a \to \bar{a}) = \left(\mathbf{P}^{(i,a)} \cdot (j \leftrightarrow \mathbf{X})\right) \wedge \left(\mathbf{P}^{(i,\bar{a})} \cdot (\bar{j} \leftrightarrow \mathbf{X})\right), \tag{13}$$

where $\mathbf{P}^{(i,l)}$ is the differentiation matrix with size $2^{n-1}\times 2^n$ that is defined as:

$$\mathbf{P}^{(i,l)} = \mathbf{M}^{(i-1)} \otimes \left[l\ \bar{l}\right] \otimes \mathbf{M}^{(n-i)}, \tag{14}$$

and $\mathbf{M}^{(w)}$ is diagonal matrix with size $2^w\times 2^w$, $\left[l\ \bar{l}\right]$ is the vector for which $l = s$ for the matrix $\mathbf{P}^{(i,a)}$ and $l = \bar{a}$ for matrix $\mathbf{P}^{(i,\bar{a})}$, and ⊗ is the Kronecker product [10].

Note that the calculation $(j \leftrightarrow \mathbf{X})$ and $(\bar{j} \leftrightarrow \mathbf{X})$ in (13) agrees with the definition of state vectors for which the function value is j and $\bar{j}$, respectively. The matrices $\mathbf{P}^{(i,a)}$ and $\mathbf{P}^{(i,\bar{a})}$ allows indicating variables with values a and $\bar{a}$, respectively. The operation AND (∧) integrates these conditions.

DPBD $\partial\phi(j \to \bar{j})/\partial x_i(a \to \bar{a})$ ($\partial\mathbf{X}(j \to \bar{j})/\partial x_i(a \to \bar{a})$) does not depend on the i-th variable [8]. Therefore, the derivative vector (13) has size of 2^{n-1}.

Consider an example for calculation of derivative vector $\partial\mathbf{X}(1\to 0)/\partial x_1(1\to 0)$ for the structure function with the truth vector $\mathbf{X} = [0\ 0\ 0\ 0\ 0\ 1\ 1\ 1]^T$ (it is the truth vector of the structure function of the system depicted in Fig. 1). According to (14), the rule for the calculation of this derivative is:

$$\partial\mathbf{X}(1 \to 0)/\partial x_1(1 \to 0) = \left(\mathbf{P}^{(1,1)} \cdot (1 \leftrightarrow \mathbf{X})\right) \wedge \left(\mathbf{P}^{(1,0)} \cdot (0 \leftrightarrow \mathbf{X})\right) = [0111]^T, \tag{15}$$

where matrices $\mathbf{P}^{(1,1)}$ and $\mathbf{P}^{(1,0)}$ are defined based on the rule (14) as:

$$\mathbf{P}^{(1,1)} = \mathbf{M}^{(0)} \otimes \left[1 \quad 0\right] \otimes \mathbf{M}^{(2)} \text{ and } \mathbf{P}^{(1,10} = \mathbf{M}^{(0)} \otimes \left[0 \quad 1\right] \otimes \mathbf{M}^{(2)}.$$

The derivative vector $\partial\mathbf{X}(1\to 0)/\partial x_2(1\to 0)$ has three nonzero values that imply that the change of the structure function value from 1 to 0 is caused by change of the first variable value from 1 to 0 if the values of the second and third variable are 1, or one of these variables has 0-value and other has value 1. In term of components states and the system availability, the DPBD indicate three state vectors $\boldsymbol{x} = (x_1, x_2, x_3)$: (<u>1→0</u>, 1, 1), (<u>1→0</u>, 0, 1) and (<u>1→0</u>, 1, 0). Therefore, the failure of the first component causes a system breakdown for working state of the second and the third components or working state of one of them. This result is equal to result that has been calculated by definition (4) for DPBD $\partial\phi(1\to 0)/\partial x_1(1\to 0)$.

A matrix procedure can be transform in parallel procedure according to [10]. Therefore the equation (13) can be interpreted by parallel procedure. For example, the flow diagrams for the calculation of the derivative vectors $\partial\mathbf{X}(1\to 0)/\partial x_1(1\to 0)$, $\partial\mathbf{X}(1\to 0)/\partial x_2(1\to 0)$and $\partial\mathbf{X}(1\to 0)/\partial x_3(1\to 0)$ for the structure function (5) according (13) are presented in Fig. 3. These diagrams illustrate the possibility to use parallel procedures for the calculation of DPBD.

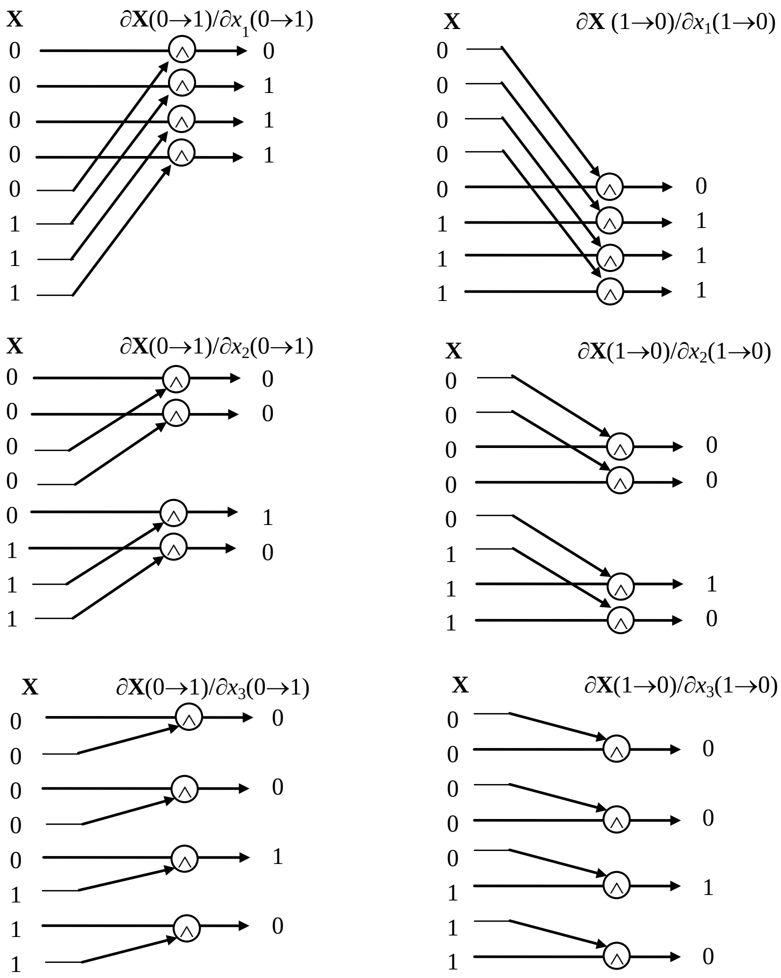


Fig. 3. Calculation of DPBDs based on parallel procedures

4 Conclusion

In this paper the new algorithm based on the parallel procedures is proposed. All most often-used IMs (Table 1) can be calculated based on this algorithm according (6) – (11). The computational complexity of the proposed algorithm is less in comparison with algorithm based on the typical analytical calculation (Fig. 4).

The proposed algorithm for the calculation of IMs based on the parallel procedures can be used in many practical applications. The principal step in these applications is representation of the investigated object by the structure function. As a rule the structure function is defined based on analysis of the structure of investigated object. For example, this algorithm can be used for calculation of importance of service points in service system in [9].

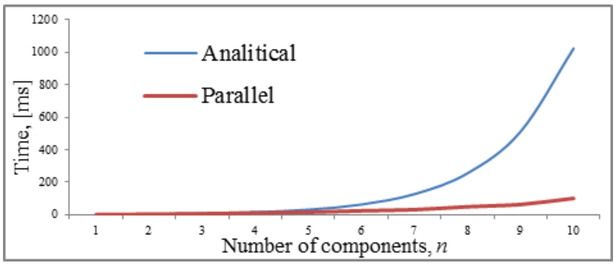


Fig. 4. Computation time for calculation of DPBDs based on analytical and parallel procedures

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